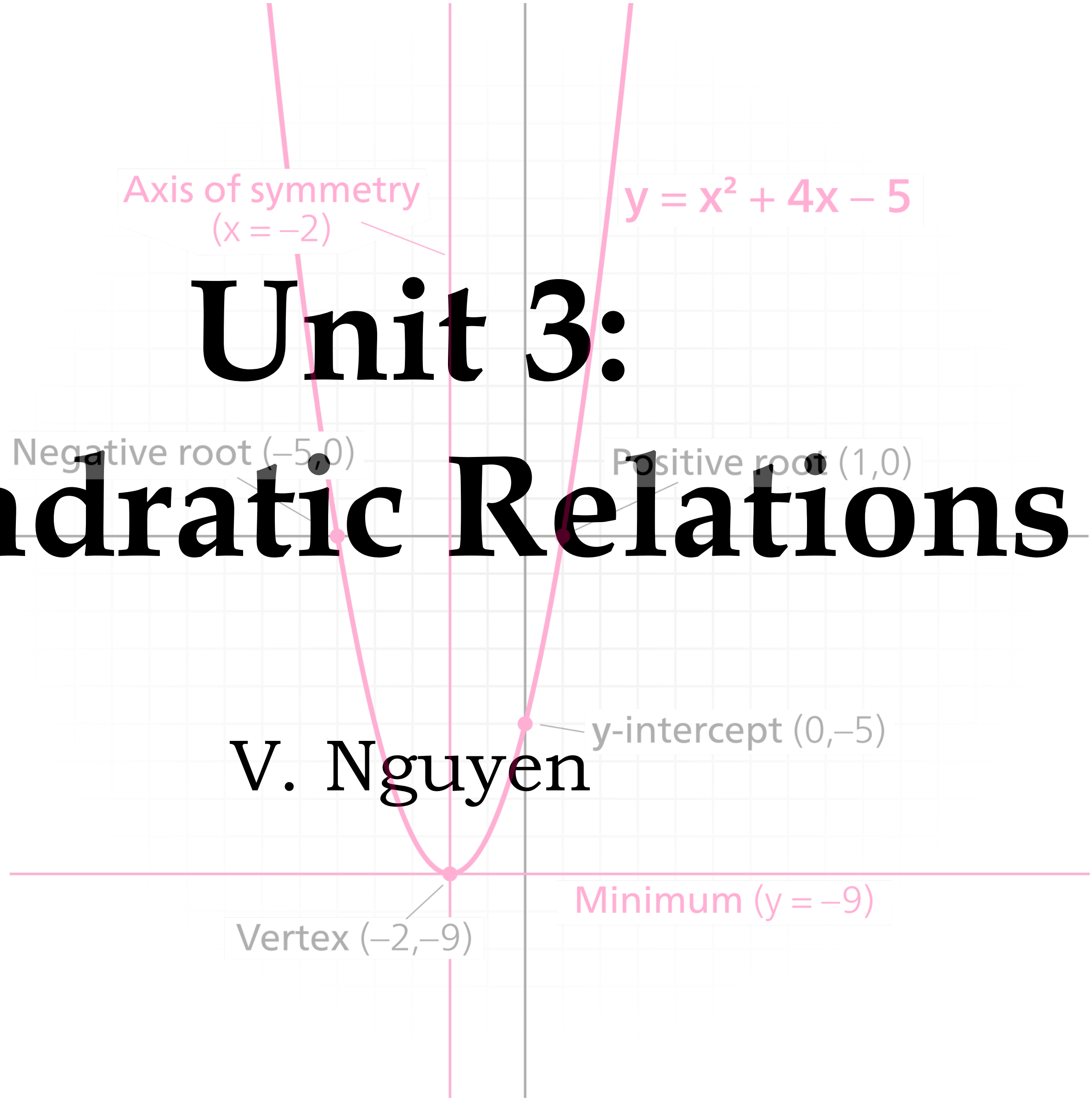


Unit 3: Quadratic Relations

V. Nguyen

Quadratic Relations

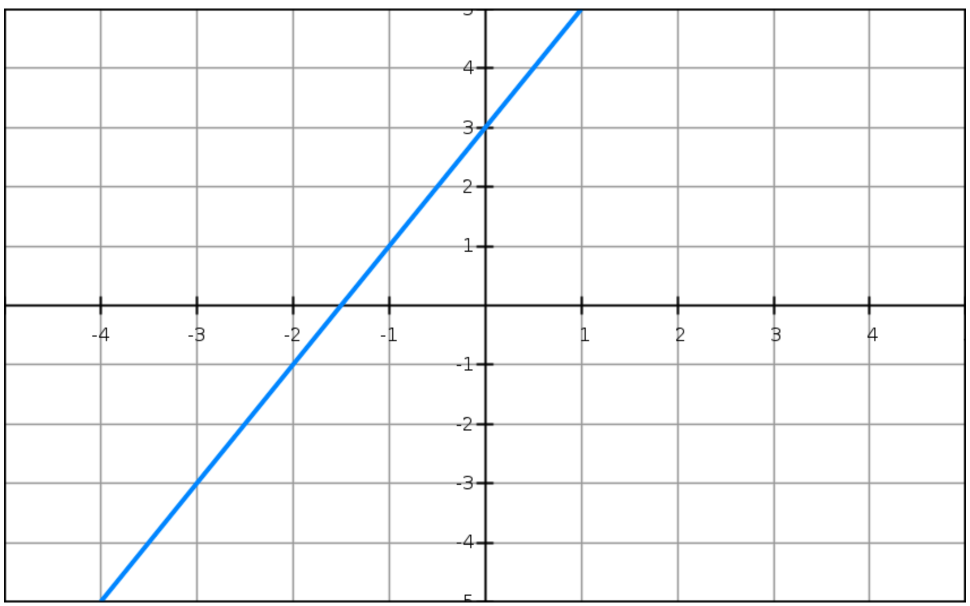
Quadratics in Standard Form

- <u>Goal:</u> Investigate graphs and properties of quadratic relations in standard form.

$$ax^2+bx+c$$

Linear Relations

A graph of a linear relation has the form: $y = mx + b$



Quadratic Relations

The graph of a quadratic relation has the form $y = ax^2 + bx + c$ is not a straight line since the value of x^2 increases much quicker than x.

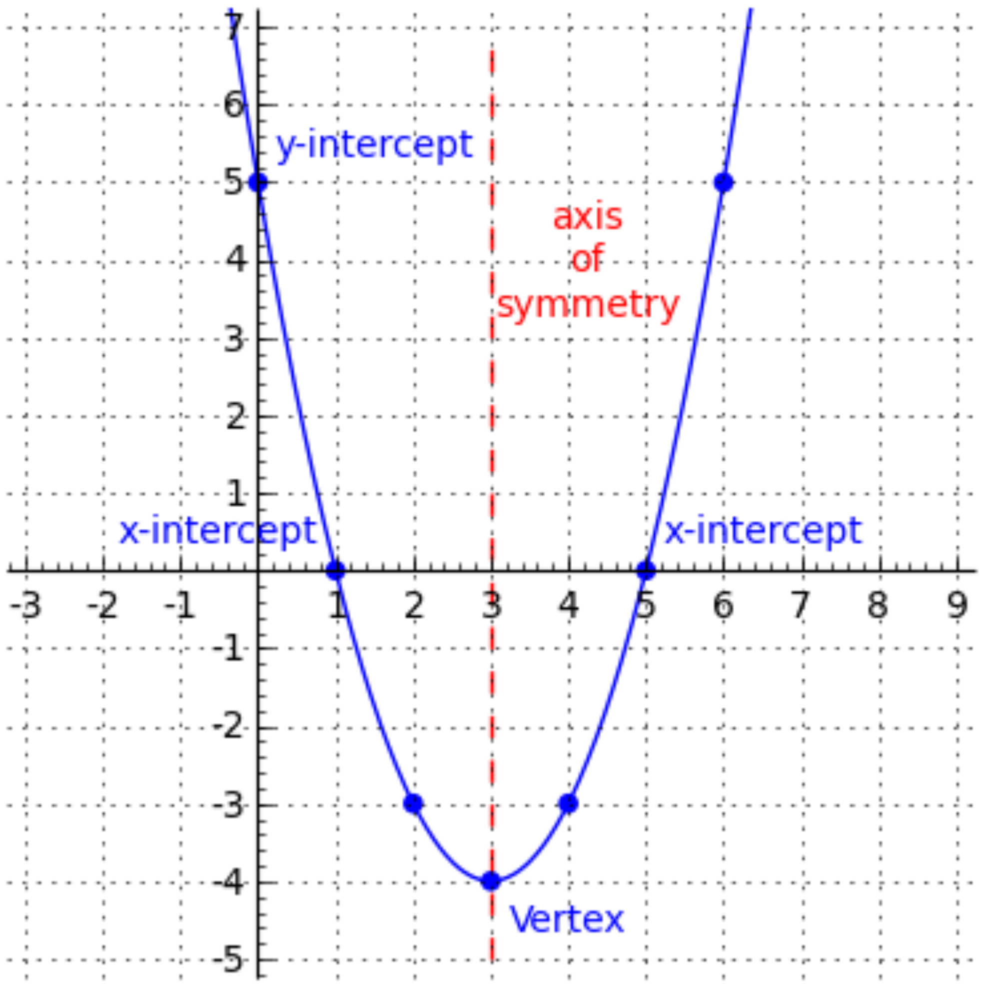
Example 1: Sketch the graph of $y = x^2 - 2x + 3$

Construct a table of values to determine points.

x	$y = x^2 - 2x + 3$
0	$0^2 - 2(0) + 3 = 3$
1	$1^2 - 2(1) + 3 = 1$
2	$2^2 - 2(2) + 3 = 3$
3	$3^2 - 2(3) + 3 = 6$
4	$4^2 - 2(4) + 3 = 11$

Quadratic Relations

Key features:



Quadratic Relations: Standard Form

<u>Example 2:</u> Sketch the graph of $y = 2x^2 - 6$ and state its key features.

x	$y =$

Quadratic Relations: Standard Form

Example 3: Sketch the graph of $y = -x^2 - 2x - 2$ and state its key features.

x	$y =$

Quadratic Relations

Recall that the **first differences** of a linear relation are constant.

A quadratic relation has a constant **second differences**.

Let's compare these two relations:

$y = 2x + 3$

x	y	
		1D

$y = x^2 - 2x + 3$

x	y		
		1D	
			2D

Quadratic Relations

Example 4: Classify the relations below as linear, quadratic or neither.

x	y		
1	-7	**1D**	
2	2	9	**2D**
3	17	15	6
4	38	21	6
5	65	27	6

x	y		
1	-4	**1D**	
2	3	7	**2D**
3	22	19	12
4	59	37	18
5	120	61	24

Quadratic Relations: Standard Form

In summary, the standard form $y = ax^2 + bx + c$ of a quadratic relation give us the following key features:

- If $a > 0$, the parabola opens upward 🙂
- If $a < 0$, the parabola opens downward ☹️
- The y-intercept is at $(0, c)$

Quadratic Relations

Quadratics in Vertex Form

- Goal: Investigate graphs and properties of quadratic relations in vertex form.

$$y=(x+6)^2-4$$

$$h=-6 \quad k=-4 \longrightarrow \text{Vertex: } (-6,-4)$$

Quadratic Relations: Vertex Form

The quadratic relation can also be in **vertex form**: $y = a(x - h)^2 + k$.
Let's determine the vertex of the following examples by using a table of values.

<u>1:</u> $y = x^2 + 3$

x	$y = x^2 + 3$
0	$(0)^2 + 3 = 0 + 3 = 3$
1	$(1)^2 + 3 = 1 + 3 = 4$
2	$(2)^2 + 3 = 4 + 3 = 7$
3	$(3)^2 + 3 = 9 + 3 = 12$
4	$(4)^2 + 3 = 16 + 3 = 19$

<u>2:</u> $y = (x - 2)^2$

x	$y = (x - 2)^2$
0	$(0 - 2)^2 = (-2)^2 = 4$
1	$(1 - 2)^2 = (1)^2 = 1$
2	$(2 - 2)^2 = (0)^2 = 0$
3	$(3 - 2)^2 = (1)^2 = 1$
4	$(4 - 2)^2 = (2)^2 = 4$

<u>3:</u> $y = (x - 2)^2 + 3$

x	$y = (x - 2)^2 + 3$
0	$(0-2)^2 + 3 = (-2)^2 + 3 = 4 + 3 = 7$
1	$(1-2)^2 + 3 = (-1)^2 + 3 = 1 + 3 = 4$
2	$(2-2)^2 + 3 = (0)^2 + 3 = 0 + 3 = 3$
3	$(3-2)^2 + 3 = (1)^2 + 3 = 1 + 3 = 4$
4	$(4-2)^2 + 3 = (2)^2 + 3 = 4 + 3 = 7$

<u>Observation:</u> Pretty long and tedious… but did you notice anything?

Quadratic Relations: Vertex Form

Looking at the previous 3 examples, we observe the following:

- The vertex of $y = x^2 + 3$ is at (0,3) – – – – – – (0 , k)
- The vertex of $y = (x - 2)^2$ is at (2,0) – – – – – – (h , 0)
- The vertex of $y = (x - 2)^2 + 3$ is at (2,3) – – – – – – (h , k)

Therefore, the quadratic relation of the form $y = a(x - h)^2 + k$ will have its vertex at (h , k).

Quadratic Relations: Vertex Form

But wait, there's more…

1: $y = x^2 + 3$

x	y		
0	3	**1D**	
1	4		**2D**
2	7		
3	12		
4	19		

2: $y = (x - 2)^2$

x	y		
0	3	**1D**	
1	4		**2D**
2	7		
3	12		
4	19		

3: $y = (x - 2)^2 + 3$

x	y		
0	3	**1D**	
1	4		**2D**
2	7		
3	12		
4	19		

Question: What did you notice about the **first difference (1D)**?

Quadratic Relations: Vertex Form

What does the graph look like for $y = (x - 2)^2 + 3$?

Step pattern:

Quadratic Relations: Vertex Form

Example 1: Graph the parabola given by $y = 2(x + 1)^2 + 5$

Quadratic Relations: Vertex Form

Example 2: Graph the parabola given by $y = -3(x - 1)^2 + 2$

Quadratic Relations: Vertex Form

Example 3: Determine a quadratic relation with its vertex at $V(-3,12)$ if it passes through the point $P(-1,4)$.

Quadratic Relations: Vertex Form

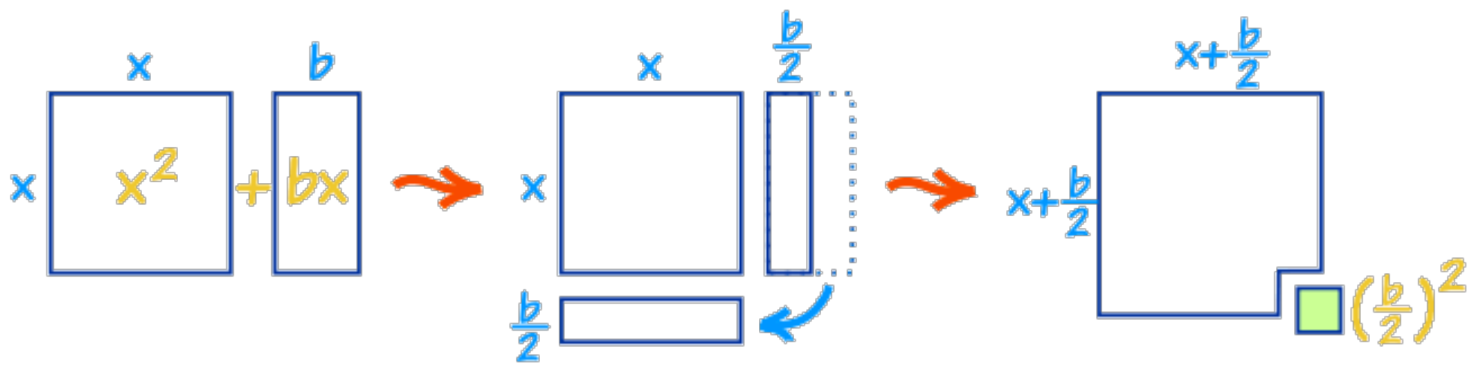
In summary, the vertex form $y = a(x - h)^2 + k$ of a quadratic relation give us the following key features:

- If $a > 0$, the parabola opens upward 🙂
- If $a < 0$, the parabola opens downward ☹️
- The *vertex* is at (h, k)
- The "**step** pattern" is given by a, $3a$, $5a$, …

Completing the Square

Part 1: Simple Trinomials

* Goal: Using algebra to convert simple quadratic functions from standard form to vertex form.



Quadratic Relations: Vertex Form

<u>Recap:</u> Sketch the graph of $y = (x + 1)^2 - 4$

Completing the Square

The purpose of completing the square (CTS) is to convert from standard form to vertex form so we can graph the function easier.

To do so, we need to rewrite a trinomial using a ________ __________.

Consider: $(x - 3)^2 + 2 = (x - 3)(x - 3) + 2 = x^2 - 6x + 9 + 2 = x^2 - 6x + 11$

But how do we go from $\underbrace{x^2 - 6x + 11}^{\textit{standard}}$ to $\overbrace{(x - 3)^2 + 2}^{\textit{vertex}}$?

Generally, we need to add and subtract a constant term of _____.

Completing the Square

<u>Example 1:</u> Convert $y = x^2 + 6x + 5$ to vertex form.

Completing the Square

Example 2: Determine the minimum value of $y = x^2 + 12x + 16$.

Completing the Square

<u>Example 3:</u> Sketch the graph of $y = x^2 - 8x + 18$.

Completing the Square

Example 3: Sketch the graph of $y = x^2 - 8x + 18$.

Completing the Square

Part 2: Complex Trinomials

- Goal: Using algebra to convert complex quadratic functions from standard form to vertex form.

$$x^2 + bx + c = 0$$
$$x^2 + bx = -c$$
$$x^2 + bx + \left(\frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2$$
$$\left(x + \frac{b}{2}\right)^2 = -c + \left(\frac{b}{2}\right)^2$$

Completing the Square

Previously, we were able to convert to vertex form when $a = 1$.

For complex trinomials, we need to ensure that the a value is factored out for the first two terms.

e.g. $3x^2 - 9x + 25 =$

$-2x^2 - 8x + 24 =$

$5x^2 - 9x + 7 =$

$-0.5x^2 - 4x + 2 =$

Completing the Square

Example 1: Convert $y = 2x^2 - 20x + 53$ to vertex form

Completing the Square

Example 2: What is the maximum value of $y = -3x^2 - 18x + 5$?

Completing the Square

Example 3: Sketch a graph of $y = 2x^2 - 8x$?

Completing the Square

Example 3: Sketch a graph of $y = 2x^2 - 8x$?

Quadratic Relations

Quadratics in Factored Form

* Goal: Determine x-intercepts for quadratic in factored form.

Distributive Law

Recall: Expand and simplify the expression

$3(x + 4)(x - 5) =$

Note: This could have also been solved by multiplying the two binomials first, then multiplying the terms of resulting trinomial by 3.

Quadratic Relations

Consider the quadratic relation $y = 2(x - 1)(x - 5)$.

Expanding the equation using distributive law gives the equivalent equation $y = 2x^2 - 12x + 10$. Using a table of values for either expression yields the same results.

x	$y = 2(x - 1)(x - 5)$
0	$2(0 - 1)(0 - 5) = 10$
1	$2(1 - 1)(1 - 5) = 0$
2	$2(2 - 1)(2 - 5) = -6$
3	$2(3 - 1)(3 - 5) = -8$
4	$2(4 - 1)(4 - 5) = -6$

x	$y = 2x^2 - 12x + 10$
0	$2(0)^2 - 12(0) + 10 = 10$
1	$2(1)^2 - 12(1) + 10 = 0$
2	$2(2)^2 - 12(2) + 10 = -6$
3	$2(3)^2 - 12(3) + 10 = -8$
4	$2(4)^2 - 12(4) + 10 = -6$

Quadratic Relations

A quadratic relation written as the product of two binomials is said to be in **factored form.**

Note: In its table of values, $y = 2(x - 1)(x - 5)$ has ***x*-intercepts** or **zeros** at (1, 0) and (5, 0). That is the **signs** of the factors appear to be ***opposite.***

Also note that the y-intercept is at (0,10), and that 2(1)(5) = 10

<u>Factored Form of a Quadratic</u>

A quadratic relation in factored form, $y = a(x - \mathrm{r})(x - \mathrm{s})$, has x-intercepts of r and s. It opens upward if $a > 0$, and downward if $a < 0$. Its y-intercept is $(a)(\mathrm{r})(\mathrm{s})$.

Quadratic Relations

Example 1: State the intercepts for the relation $y = 2(x - 3)(x - 5)$

Example 2: State the intercepts for the relation $y = -3x(x + 7)$

Quadratic Relations

Recall that a parabola's vertex lies on its axis of symmetry (AoS).

The axis is **midway** between any two points that have the same y-coordinate, such as the x-intercepts.

The x-coordinate of the vertex, then, is the average of the x-coordinates of the x-intercepts.

Factored Form of a Quadratic

The x-coordinate of the vertex, (h,k) of a parabola described by the relation $y = a(x - \mathrm{r})(x - \mathrm{s})$, is at $h = (\mathrm{r} + \mathrm{s})/2$. The y-coordinate is at $k = a(h - r)(h - s)$.

Quadratic Relations

Example 3: Determine the coordinates of the vertex of the parabola described by $y = 3(x + 2)(x - 4)$.

Quadratic Relations

Example 4: Graph the quadratic relation $y = -(x - 3)(x + 5)$

Quadratic Relations

Example 4: Graph the quadratic relation $y = -(x - 3)(x + 5)$

Quadratic Relations

Example 5: Determine an equation for a quadratic relation with x-intercepts at (7,0) and (-1,0), if it passes through (2,30).

Quadratic Relations

Converting Quadratic Relations to Factored Form

* Goal: Convert quadratic relations from standard form to factored form.

Quadratic Relations

Recall: Graph the relation $y = 2(x + 7)(x - 1)$

Converting to Factored Form

A quadratic relation may be converted to factored form using any of the the factoring techniques earlier.

Example 1: Determine the coordinates of the vertex of the relation $y = x^2 - 8x - 9$

Converting to Factored Form

Example 2: Determine the maximum value of $y = -x^2 - 8x + 33$

Converting to Factored Form

Example 3: Graph the relation $y = 2x^2 - 4x - 30$.

Converting to Factored Form

Example 4: Graph the relation $y = x^2 - 4x + 5$

Quadratic Relations

Partial Factoring

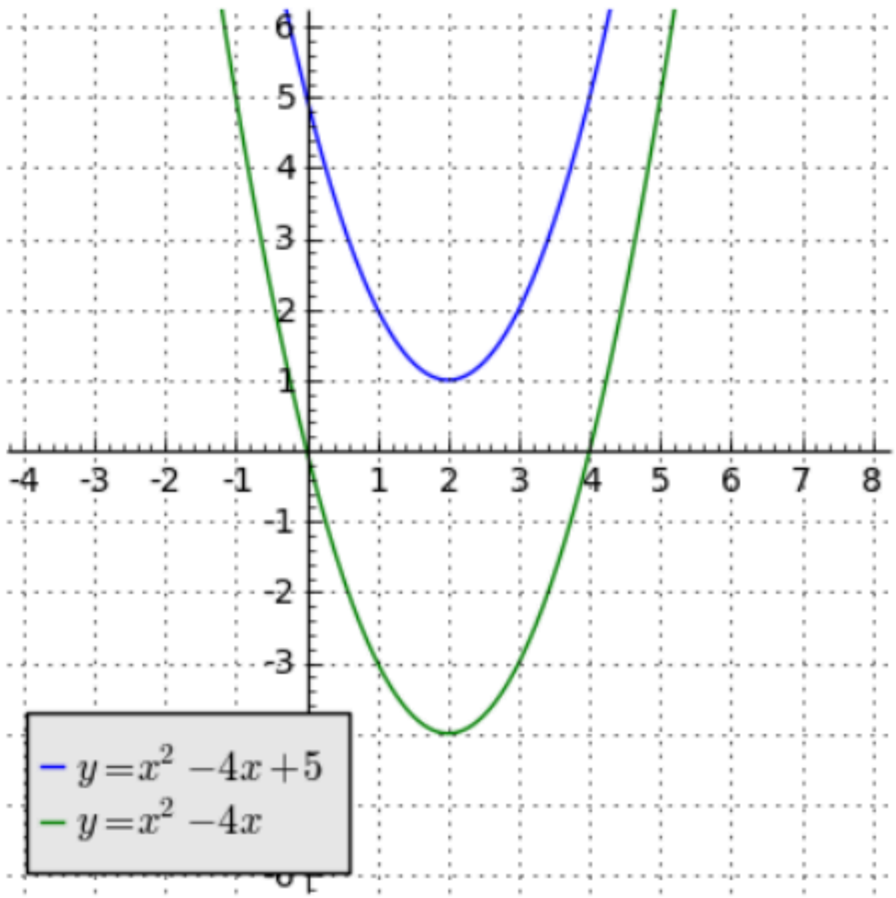
* Goal: Use partial factoring method to determine vertex of a relation.

Partial Factoring

Recall: Determine the location of the vertex for $y = 3(x + 7)(x - 1)$ by factoring.

Partial Factoring

Consider the graphs of $y = x^2 - 4x + 5$ and $y = x^2 - 4x$ below.



<u>Observations:</u>

Partial Factoring

<u>Example 1:</u> Determine the location of the vertex for $y = x^2 - 6x + 10$

Graph of $y = x^2 - 6x + 10$:

Note:

Partial Factoring

<u>Example 2:</u> Determine the location of the vertex for $y = 2x^2 + 16x + 27$

Partial Factoring

Graph of $y = 2x^2 + 16x + 27$:

Note:

Partial Factoring

Example 3: Sketch the graph of $y = -2x^2 + 20x - 37$

Partial Factoring

Example 3: Sketch the graph of $y = -2x^2 + 20x - 37$

Quadratic Relations

Applications of Quadratic Relations

* Goal: Use previously learned methods to solve for maximum or minimum value in Applications related problems.

Applications of Quadratic Relations

Many applications of quadratic relations involve finding the **optimal value** (i.e. minimum or maximum value).

For example, the maximum height of a toy rocket can be calculated by modelling its flight path with a quadratic relation and determining the location of the **vertex**.

These problems are often referred to as "**max/min**" problems.
Most of the time, "greatest", "least", "largest", "smallest", "optimal", etc. are terms that are often used to indicate the maximum or minimum value in the problem.

<u>Recall:</u> To determine the location of the vertex, we can either:

______________________________ or use __________________________.

Applications of Quadratic Relations

Example 1:

A firework, launched into the air with a velocity of 58.8 m/s from a height of 2 m, explodes at its highest point. Its height, h metres, is given by $h = -4.9t^2 + 58.8t + 2$, where t is the time in seconds. When does the firework explode? How high is it?

Applications of Quadratic Relations

Example 1:

Applications of Quadratic Relations

Example 2:

Determine the values of two numbers, one 10 greater than another, if the sum of their squares is a minimum. What is the minimum?

Applications of Quadratic Relations

Example 2:

Applications of Quadratic Relations

Example 3:

A T-shirt manufacturers typically sells 500 shirts per week to distributors for \$4.00 each. For each \$0.50 reduction in price, she estimates she can sell an additional 20 shirts. How much should she charge per shirt to maximize her revenue?

Applications of Quadratic Relations

Example 3:

Quadratic Equations

and Applications

* Goal: Use a variety of methods to solve for equations and Applications related problems.

Solving Quadratic Equations

Part 1: Solving by Factoring

<u>Warm up:</u>

If a multiply b is equal to 0, what can we say about a and b ?

What can we said about $ab = k$, if k is not equal to 0?
For example, $ab = 9$.

Solving Quadratic Equations

Part 1: Solving by Factoring

What about quadratic equations like $(x - 2)(x - 5) = 0$?

<u>Observation:</u> If a quadratic relation can be expressed in factored form, such that the product of the two binomials is ________, then we can solve for the values of ___.

Solving Quadratic Equations

Part 1: Solving by Factoring

Example 1: Solve* $x^2 - 4x - 12 = 0$

*Remember to always check with DESMOS.

Solving Quadratic Equations

Part 1: Solving by Factoring

Example 2: Solve* $2x^2 + 5x - 12 = 0$

Solving Quadratic Equations

Part 1: Solving by Factoring

Example 3: Solve* $3x^2 - 33x + 84 = 0$

Solving Quadratic Equations

Part 1: Solving by Factoring

Example 4: Solve* $4x^2 - 9 = 0$ in two ways.

Solving Quadratic Equations

Part 1: Solving by Factoring

Example 5: Solve* $x^2 - 10x = -25$

Solving Quadratic Equations

Part 1: Solving by Factoring

Example 6: Solve* $2x^2 + 5x - 9 = 6 - 2x$

Solving Quadratic Equations

Part 1: Solving by Factoring

Example 7: Solve* $4x^2 + 19x - 2 = 3x^2 + 5x + 7$

Solving Quadratic Equations

Part 1: Solving by Factoring

SUMMARY of factoring techniques in this lesson:

1. Product (ac) & Sum (b):

- For simple trinomials, i.e. $x^2 + \mathbf{b}x + \mathbf{c}$

— Find two integers whose **product** is **c** and **sum** is **b**.

- For complex trinomials, i.e. $\mathbf{a}x^2 + \mathbf{b}x + \mathbf{c}$

— Find two integers whose **product** is **ac** and **sum** is **b**.

2. Difference of Squares:

$x^2 - y^2 = (x + y)(x - y)$, e.g. $4x^2 - 49 = (2x + 7)(2x - 7)$

Solving Equations Practice

- ☐ 1. When Is a Wrestler "King of the Ring"?
- ☐ 2. What Happened When the Boarding House Blew Up?
- ☐ 3. What did Mrs. Zling say when Mr. Zling said he was going mountain climbing in the himalayas?
- ☐ 4. How Can Fishermen Save Gas?
- ☐ 5. What Do You Call a Sore on Police Officer's Foot?
- ☐ 6. Old Lawyers Never Die, They Just ________________
  Old Skiers Never Die, They Just ____________________

Solving Quadratic Equations

Part 2: Solving by Completing the Square

Example 1: Solve* $(x - 8)^2 = 7$

*Note: These questions involve "roots", more specifically square roots. Your answer must be in **root format**, i.e. $\sqrt{\#}$. **No decimals**!

Solving Quadratic Equations

Part 2: Solving by Completing the Square

Example 2: Solve* $x^2 + 8x + 8 = 0$

* Your answer must be in **root format**, i.e. $\sqrt{\#}$. **No decimals**!

Solving Quadratic Equations

Part 2: Solving by Completing the Square

Example 3: Solve* $-2x^2 + 4x + 9 = 0$

***No decimals**! $\sqrt{\#}$ format always.

Solving Quadratic Equations

Part 2: Solving by Completing the Square

Example 4: Solve* $3x^2 - 12x + 16 = 0$

***No decimals**! $\sqrt{\#}$ format always.

Solving Quadratic Equations

Part 3: Solving by using The Quadratic Formula

Consider the general (standard) form of a quadratic relation, $\boldsymbol{y = ax^2 + bx + c}$. We can complete the square to determine values of x for which $\boldsymbol{ax^2 + bx + c = 0}$.

Solving Quadratic Equations

Part 3: Solving by using The Quadratic Formula

The Quadratic Formula: $$x = -\frac{-b \pm \sqrt{b^2 \; - 4ac}}{2a}$$

Solving Quadratic Equations

Part 3: Solving by using The Quadratic Formula

Example 1: Verify that $2x^2 - 12x + 12 = 0$, has solutions $x = 3 \pm \sqrt{3}$ using the quadratic formula.

Solving Quadratic Equations

Part 3: Solving by using The Quadratic Formula

Example 2: Solve $x^2 - 6x - 91 = 0$ using the quadratic formula.

Solving Quadratic Equations

Part 3: Solving by using The Quadratic Formula

Example 3: Determine the x-intercepts of the parabola defined by $y = -3x^2 - 6x + 15$.

Solving Quadratic Equations

Part 3: Solving by using The Quadratic Formula

Example 4: Solve $2x^2 - x + 5 = 0$.

Solving Quadratic Equations

Part 3: Solving by using The Quadratic Formula

The Discriminant

Let $\boldsymbol{D = b^2 - 4ac}$, the expression inside the radical ($\sqrt{\ }$) sign in the quadratic formula. The expression is known as the *discriminant*, and it can be used to give us information about both the number of **real**-valued **solutions** for a quadratic equation.

The number of **real solutions (roots)** to a quadratic equation can be determined using the discriminant, $\boldsymbol{D = b^2 - 4ac}$.

- If $D < 0$, there are **no** real solutions.
- If $D = 0$, there is **one** real solution.
- If $D > 0$, there are **two distinct** real solutions.

Solving Quadratic Equations

Part 3: Solving by using The Quadratic Formula

Example 5: Determine the number of real solutions to $4x^2 - 20x + 25 = 0$.

Example 6: Determine the number of real solutions to $3x^2 + 5x + 4 = 0$.